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Stationary Boltzmann Equation for a Degenerate Gas in a Slab: Boundary Value Problem and Hydrodynamics

N. Ianiro[1] **and L. Triolo**[2]

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The boundary value problem for the stationary Boltzmann equation for a model gas in a plane slab is solved in full generality. The asymptotic behavior as the size of the slab goes to infinity is studied via a Chapman–Enskog expansion.

KEY WORDS: Boltzmann equation; stationary solutions; Chapman–Enskog expansion.

1. INTRODUCTION

We consider the Boltzmann equation describing the stationary state of a rarefied gas confined in a slab where the molecules undergo very special collisions. The boundary conditions are of diffusive type: when a particle hits the wall it is reemitted with a given distribtion. In this way we construct a nonequilibrium model of a rarefied gas and we investigate its stationary distributions and its hydrodynamic behavior. Because of the simplified interaction, the analysis leads to rather detailed information on the system and gives some insight into the more physical situations which have been the object of many investigations.[1-3]

The two-dimensional collision model is the following: if (v_x, v_y) and (v_{1x}, v_{1y}) are the incoming velocities, we assume that the outgoing ones are simply given by the rule

$$\begin{aligned} v'_x &= v_{1x}, \qquad v'_y = v_y \\ v'_{1x} &= v_x, \qquad v'_{1y} = v_{1y} \end{aligned} \tag{1.1}$$

[1] Dipartimento di Matematica Pura ed Applicata, Universitá dell'Aquila, Aquila, Italy.
[2] Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate, Università "La Sapienza," Rome, Italy.

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i.e., exchange in the x direction and conservation in the y direction (we think of match sticks parallel to the y axis).

We shall study the system in the Boltzmann–Grad limit: i.e., the description is reduced to the kinetic level. The corresponding Boltzmann equation for the distribution function $f_t(x, y, v_x, v_y)$ can be written in a straightforward way; putting equal to one the constant in the collision integral, we have

$$\begin{aligned}(\partial_t + v_x \partial_x + v_y \partial_y) f_t(x, y, v_x, v_y) &\\ = \int |v_x - v_{1x}| \{ f_t(x, y, v_{1x}, v_y) f_t(x, y, v_x, v_{1y}) &\\ - f_t(x, y, v_x, v_y) f_t(x, y, v_{1x}, v_{1y}) \} \, dv_{1x} \, dv_{1y} &\end{aligned} \tag{1.2}$$

2. STATIONARY BOUNDARY VALUE PROBLEM: RESULTS ON THE SOLUTION

We consider the particles confined in a slab with vertical walls at $x = -L$, $x = L$. We shall look for a stationary solution that depends only on $(x, v_x, v_y) \in [-L, L] \times \mathbb{R} \times \mathbb{R}$. The boundary conditions will be given in a slightly more general way than in the usual case, where heat exchange with reservoirs at given temperatures is assumed to occur at the walls.

The boundary conditions at $x = \pm L$ are

$$\begin{aligned} f(-L, v_x, v_y) &= -H_-(v_x, v_y) \int_{v_x \leqslant 0} v_x f(-L, v_x, v_y) \, dv_x \, dv_y, \qquad v_x \geqslant 0 \\ f(L, v_x, v_y) &= H_+(v_x, v_y) \int_{v_x \geqslant 0} v_x f(L, v_x, v_y) \, dv_x \, dv_y, \qquad v_x \leqslant 0 \end{aligned} \tag{2.1}$$

where H_- and H_+ are nonnegative functions such that

$$\begin{aligned} \int_{v_x \leqslant 0} v_x H_+(v_x, v_y) \, dv_x \, dv_y &= -1 \\ \int_{v_x \geqslant 0} v_x H_-(v_x, v_y) \, dv_x \, dv_y &= +1 \end{aligned} \tag{2.2}$$

Such conditions take care of the requirement that the net mass flux across each boundary is zero.

The equation (1.2) will be reduced to the following form:

$$\begin{aligned} v_x \partial_x f(x, v_x, v_y) = \int |v_x - v_{1x}| \{ f(x, v_{1x}, v_y) f(x, v_x, v_{1y}) &\\ - f(x, v_x, v_y) f(x, v_{1x}, v_{1y}) \} \, dv_{1x} \, dv_{1y} &\end{aligned} \tag{2.3}$$

We remark that when H_- and H_+ are given by the single function H,

$$H(v_x, v_y) = \begin{cases} H_+(v_x, v_y) = \phi_+(v_x)\, \psi(v_y), & v_x \leqslant 0 \\ H_-(v_x, v_y) = \phi_-(v_x)\, \psi(v_y), & v_x \geqslant 0 \end{cases}$$

i.e., factorization holds (since the v_y distribution ψ is the same at the two walls), then the stationary solution of (2.1), (2.3) is given by

$$f(x, v_x, v_y) = H(v_x, v_y) n \Big/ \int H(v_x, v_y)\, dv_x\, dv_y$$

where $n = \int f(x, v_x, v_y)\, dv_x\, dv_y$ represents the number density; it is constant because from Eq. (2.3) the "partial density" $\int f(x, v_x v_y)\, dv_y$ is already independent of x. Let us fix the total number of particles $2Ln = \int f(x, v_x v_y)\, dv_y\, dv_x\, dx$.

In the general case we have the following results:

Theorem 1 (Existence and uniqueness). For any functions $H_\pm$ satisfying (2.2) there exists a unique, stationary solution $f_L(x, v_x, v_y)$ for the problem (2.1), (2.3). This solution is bounded in the sup norm in term of the boundary data.

To prove Theorem 2, we need to restrict the boundary distributions as follows:

$$\begin{aligned} H_+(v_x, v_y) &= \phi_+(v_x)\, \psi_+(v_y), \qquad v_x \leqslant 0 \\ H_-(v_x, v_y) &= \phi_-(v_x)\, \psi_-(v_y), \qquad v_x \geqslant 0 \end{aligned} \tag{2.4}$$

(this assumption corresponds to the choice of "Maxwellian" outgoing distribution for more physical models). We can choose $\int \psi_\pm(v_y)\, dv_y = 1$ and, if $\chi(\cdot)$ denotes an indicator function, posing

$$\phi(v_x) = \phi_+(v_x)\, \chi(v_x \leqslant 0) + \phi_-(v_x)\, \chi(v_x \geqslant 0)$$

we have:

Theorem 2 (Asymptotic behavior). For any $H_\pm$ satisfying (2.2) and (2.4) we have

$$f_L(x, v_x, v_y) \sim \frac{n\phi(v_x)}{\int \phi(v_x)\, dv_x} \left(\frac{\psi_+(v_y) + \psi_-(v_y)}{2} + \frac{\psi_+(v_y) - \psi_-(v_y)}{2L} x \right) \qquad \text{as} \quad L \to \infty$$

Theorem 1 is proven in the next section by an iterative procedure.

From Theorem 2 we have that when the size of the region goes to infinity, the stationary solution will be very close to the equilibrium one plus a linear term with a gradient proportional to $1/L$.

We prove this asymptotic behavior by using the Chapman–Enskog procedure.[4] Associated with the kinetic equation we will introduce a stochastic process that gives a simpler formulation of the previous problem. In the last section we consider the same model in the case where the velocities are just ± 1; in this case the solution is given explicitly.

3. THE EXISTENCE THEOREM

Integrating Eq. (2.3) on the v_y variable, we have

$$\int f(x, v_x, v_y)\, dv_y = g_f(v_x), \qquad v_x \neq 0$$

i.e., "the partial density" $g_f(v_x)$ does not depend on x for a.e. v_x. We can evaluate it in the following way: if $J_\pm$ denote the half-moments on the function $g_f(v_x)$,

$$J_+ = \int_{v_x \geqslant 0} v_x\, g_f(v_x)\, dv_x, \qquad J_- = \int_{v_x \leqslant 0} v_x\, g_f(v_x)\, dv_x$$

then, using the normalization conditions, we get

$$J_+ = -J_- = J = n \left(\int_{v_x \geqslant 0} H_-\, dv_x\, dv_y + \int_{v_x \leqslant 0} H_+\, dv_x\, dv_y \right)^{-1}$$

From the boundary conditions, integrating over v_y, we obtain

$$\begin{aligned} g(v_x) &= J \int H_-(v_x, v_y)\, dv_y, \qquad v_x > 0 \\ g(v_x) &= J \int H_+(v_x, v_y)\, dv_y, \qquad v_x < 0 \end{aligned} \tag{3.1}$$

The function $g(v_x)$ depends on the boundary distribution functions up to the normalization constant J, but not on the distribution function itself: this transforms our problem into a linear one.

Let us call $\mathbf{K}[[\phi]$ the following integral operator:

$$\mathbf{K}[\phi](u) = \int |u - u'|\, \phi(u')\, du'$$

Then Eq. (2.3) becomes

$$v_x \partial_x f(x, v_x, v_y) = g(v_x)\, \mathbf{K}[f](x, v_x, v_y) - f(x, v_x, v_y)\, k(v_x) \tag{3.2}$$

where action of the operator on the v_x variable is understood and $k(v_x) := \mathbf{K}[g](v_x)$.

In order to study the linear problem we have obtained, let us change notations slightly, and also take a different approach to the various questions involved.

Since the component v_y appears in the equation only as a parameter, as in ref. 5 we can consider it as a "color" specification, denoting it by c, and we use the shorter symbol v for v_x; in this way the following interpretation of the equation clearly emerges: it is the stationary kinetic equation associated with the evolution of independent test particles (indexed by c) in the interval $[-L, L]$, which interact with a background medium [described by the distribution $g(v)$]; this interaction is of collisional type and the collisional frequency is given by $\mathbf{K}[g]$.

For a fixed value of c, the boundary conditions no longe give (in general) mass conservation: the walls act as absorbers and/or producers of particles of that "color" c, with possible unbalancing of mass.

A complete study of the problem would give the function $f(x, v, c)$ for every c from the profile of the distribution function $h(x, v \mid c)$ through the relation $h(x, v \mid c) = f(x, v, c)/g(v)$. In order to have a notation suitable for the following theorem, too, we pass to a scaled space variable: we set $s = x/L$, $-1 \leqslant s \leqslant 1$, and $h(s, v \mid c) = f(sL, v, c)/g(v)$; calling $\varepsilon = 1/L$, we have for Eq. (3.2)

$$v\, \partial_s h = \varepsilon^{-1} A h \tag{3.3}$$

where

$$A\phi(v) = \int |v - v'|\, g(v') \{\phi(v) - \phi(v)\}\, dv'$$

is an unbounded operator defined on the dense set

$$D(A) = \left\{ \phi \in \mathscr{L}_2(\mathbb{R}, g(v)\, dv) \colon \int v^2 \phi^2(v)\, g(v)\, dv < \infty \right\}$$

and the boundary conditions are given at $s = \pm 1$ as

$$\begin{aligned} h(-1, v \mid c) &= JH_-(v, c)/g(v), \qquad v > 0 \\ h(1, v \mid c) &= JH_+(v, c)/g(v), \qquad v < 0 \end{aligned} \tag{3.4}$$

We note that $Ah = 0$ iff h is independent of v, so that when $H_\pm(v, c) = g(v)\, \psi(c)/J$ (where plus stands for $v > 0$ and minus stands for $v < 0$), then the equilibrium solution is $h(x, v \mid c) = \psi(c)$.

The interpretation of Eq. (3.3) of use for the forthcoming analysis comes from the original particle model. To know the color of a particle, one simply has to follow it back until the particle comes out from the wall. At that time its color (v_y velocity) distribution is determined by its $v = v_x$ velocity. Therefore we consider the stochastic process on a state space $A = \{|s| < 1,\ v \in \mathbb{R}\}$ having the generator G, which acts on bounded and $\mathscr{C}^1$ function h,

$$Gh(s, v \mid c) = -v\, \partial_s h(s, v \mid c) + \int |v - v'|\, g(v')\{h(s, v' \mid c) - h(s, v \mid c)\}\, dv'$$

Namely, the v part of the process is a jump process with rate $k(v)$ and transition probability density $|v - v'|\, g(v')/k(v)$. The space part is just a drift with rate $-v$ (the minus sign comes from the fact that we are looking at the backward equation for the process). We stop the process at the first time τ it reaches the boundaries $s = \pm 1$ (with velocities $v < 0$, $v > 0$, respectively). Notice that τ is finite with probability one, as can easily be seen.

This interpretation suggests that we write the integral equation for the evolution associated with the previously considered process[6,7]

$$\begin{aligned} h_t(s, v \mid c) = {} & h_0(s, v \mid c) \exp[-\varepsilon^{-1} k(v) t]\, \chi(t \leqslant \zeta_A(s, v)) \\ & + h^*(s_{\zeta_A}, v_{\zeta_A} \mid c)\, \chi(t > \zeta_A(s, v)) \exp[-\varepsilon^{-1} k(v)\, \zeta_A(s, v)] \\ & + \int_0^{t \wedge \zeta_A} \exp[-\varepsilon^{-1} k(v) \sigma]\, \varepsilon^{-1} k(v) \\ & \times \int \frac{|v - w|\, g(w)}{k(v)}\, h_{t-\sigma}(s - v\sigma, w \mid c)\, dw\, d\sigma \end{aligned} \tag{3.5}$$

where $\zeta_A(s, v) = [s + \mathrm{sign}(v)]/v$ is the time it takes a particle, starting at (s, v), for the first crossing of the boundary without suffering a jump in velocity, and $h^*(s, v \mid c)$ for $s = \pm 1$ gives the boundary values. Passing to the limit $t \to +\infty$ and stressing the dependence on the parameter ε, we obtain the stationary integral equation corresponding to Eq. (3.3):

$$\begin{aligned} h^\varepsilon(s, v \mid c) = {} & h^B(v \mid c) \exp\left[-\varepsilon^{-1} k(v) \frac{\mathrm{sign}(v) + s}{v}\right] \\ & + \int_0^{[\mathrm{sign}(v) + s]/v} \varepsilon^{-1} \exp[-\varepsilon^{-1} k(v) \sigma] \\ & \times \int |v - v'|\, g(v')\, h^\varepsilon(s - v\sigma, v' \mid c)\, dv'\, d\sigma \end{aligned} \tag{3.6}$$

where

$$h^B(v \mid c) = h^*(-1, v \mid c)\,\chi(v > 0) + h^*(1, v \mid c)\,\chi(v < 0)$$

is the given "boundary" function.

This equation could have been obtained via the change of variables $y = s - v\sigma$ in the integral term treated as a perturbation of the homogeneous equation $v\partial_s \psi + \varepsilon^{-1} k(v)\psi = 0$; in this way one could obtain an integral equation similar to the one worked out for the Boltzmann boundary value problem.[8-10]

Now we shall prove through an iterative procedure the existence and uniqueness of the solution of Eq. (3.6); moreover, the solution is bounded by the sup-norm of the boundary datum. This is fairly obvious from the point of view of the stochastic interpretation, since the solution is the expected value of the boundary datum h^B in the following sense:

$$h(s, v \mid c) = \mathbb{E}_{s,v}(h^B(v_{\tau_A} \mid c))$$

where τ_A is the random time of crossing the boundary starting from $(s, v) \in A$ (along the process described above). In fact this is readily achieved by studying the sequence (where dependence on the parameter c is omitted)

$$\begin{aligned} h^{(N+1)}(s, v) = {} & h^B(v) \exp\left[-\varepsilon^{-1} k(v) \frac{\operatorname{sign}(v) + s}{v} \right] \\ & + \int_0^{[\operatorname{sign}(v) + s]/v} \varepsilon^{-1} \exp[-\varepsilon^{-1} k(v)\sigma] \\ & \times \int |v - w|\, g(w)\, h^{(N)}(s - v\sigma, w)\, dw\, d\sigma, \quad N \geqslant 1, \;\; h^{(0)}(s, v) = 0 \end{aligned} \tag{3.7}$$

We have

$$|h^{(1)}(s, v)| = \left| h^B(v) \exp\left[-\varepsilon^{-1} k(v) \frac{\operatorname{sign}(v) + s}{v} \right] \right| \leqslant \|h^B\|_\infty$$

Now if we suppose that $\|h^{(N)}\|_\infty \leqslant \|h^B\|_\infty$, we can show that $\|h^{(N+1)}\|_\infty \leqslant \|h^B\|_\infty$:

$$\begin{aligned} |h^{(N+1)}(s, v)| \leqslant {} & \|h^B\|_\infty \exp\left[-\frac{s + \operatorname{sign}(v)}{v} \varepsilon^{-1} k(v) \right] \\ & + \int_0^{[s + \operatorname{sign}(v)]/v} \exp[-\varepsilon^{-1} k(v)\sigma]\, \varepsilon^{-1} \\ & \times \int |v - w|\, g(w)\, \|h^B\|_\infty\, dw\, d\sigma \end{aligned}$$

$$= \|h^B\|_\infty \exp\left[-\frac{s+\text{sign}(v)}{v}\varepsilon^{-1}k(v)\right]$$
$$+ \|h^B\|_\infty \left\{1 - \exp\left[-\frac{s+\text{sign}(v)}{v}\varepsilon^{-1}k(v)\right]\right\}$$
$$= \|h^B\|_\infty$$

(It is to be noted for later considerations that this estimate is uniform in ε.)

This iteration leads to existence and uniqueness of the solution through a monotonicity argument, at least if we start from a nonnegative boundary datum h^B, and this is reasonable because $h(s, v \mid c) := f(s, v, c)/g(v)$, where both f and g are nonnegative. In fact, by subtracting one step from the preceding one, we have

$$[h^{(N+1)} - h^{(N)}](s, v) = \varepsilon^{-1} \int_0^{[s+\text{sign}(v)]/v} \exp[-\varepsilon^{-1}k(v)\sigma]$$
$$\times \int |v-w|\, g(w)[h^{(N)} - h^{(N-1)}](s - v\sigma, w)\, dw\, d\sigma$$

so, if $[h^{(N)} - h^{(N-1)}](s, v) \geqslant 0$, the same will hold for $[h^{(N+1)} - h^{(N)}](s, v)$; and, since we suppose $h^B \geqslant 0$, it follows that

$$[h^{(1)} - h^{(0)}](s, v) = h^{(1)}(s, v) = h^B(v) \exp\left[-\varepsilon^{-1}k(v)\frac{s+\text{sign}(v)}{v}\right] \geqslant 0$$

Hence, there exists

$$\lim_{N\to\infty} h^{(N)}(s, v) = h^\varepsilon(s, v) \qquad \text{and} \qquad \|h^\varepsilon\|_\infty \leqslant \|h^B\|_\infty$$

Moreover, the limit function satisfies Eq. (3.6), as can easily be seen. From the equation it follows that $v\partial_s h$ exists, so that Eq. (3.3) is satisfied; uniqueness is obtained by applying the last estimate to the difference of two possible solutions (with the same h^B): it satisfies the equation with zero boundary value.

4. ASYMPTOTIC HYDRODYNAMIC BEHAVIOR

We consider now the asymptotic of the solution as $\varepsilon \to 0^+$. Looking at the associated process, we can imagine that when the size L goes to infinity, the test particle that starts from a fixed (s, v) suffers so many jumps before reaching the boundary that it eventually becomes "thermalized": namely, it will lose memory of the initial velocity and its velocity distribution will become very close to the equilibrium one. By the invariance principle its motion (suitably rescaled) converges to a Brownian motion, so that the probability of being at $\pm L$ will depend linearly on the initial position x. In this original microscopic variable $x \in [-L, L]$ the gradient is proportional to $\varepsilon = 1/L$ and to the difference of the boundary distribution for the given color c. It would be possible to prove this rigorously, but we prefer to use a more physical argument, i.e., an adaptation of the Chapman–Enskog procedure.[4]

We should like to show that the solution has the form

$$h^\varepsilon(s, v \mid c) = h_0(s, v \mid c) + O(\varepsilon)$$

where $h_0(s, v \mid c) = h_0(s \mid c)$ (the dependence on v in the corresponding distribution function $f_0 := h_0 \cdot g$ is in the factor g, which represents the equilibrium distribution); and where h_0 is given by the sum of a global equilibrium term plus a linear one, with a gradient given by the differences of the boundary color distribution. This will be shown in the following. We start from Eq. (3.3),

$$v\, \partial_s h^\varepsilon = \varepsilon^{-1} \Lambda h^\varepsilon$$

with boundary conditions for the distribution function $f^\varepsilon(s, v, c) = h^c(x, v \mid c) \cdot g(v)$ "factorized," i.e., given as a product of functions of v and c. This means that for the function h^ε we get boundary values dependent on c only:

$$h^\varepsilon(+1, v \mid c) = \psi_+(c), \quad v < 0; \qquad h^\varepsilon(-1, v \mid c) = \psi_-(c), \qquad v > 0$$

We assume that the solution h^ε has the form

$$h^\varepsilon = h_0(s \mid c) + \varepsilon h_1(s, v \mid c) + R^\varepsilon(s, v, c)$$

so that Eq. (3.3) becomes

$$v\, \partial_s(h_0 + \varepsilon h_1 + R^\varepsilon) = \varepsilon^{-1} \Lambda(h_0 + \varepsilon h_1 + R^\varepsilon) \tag{4.1}$$

By equating coefficients of equal powers of ε, we have

$$\Lambda h_0 = 0, \qquad v\, \partial_s h_0 = \Lambda h_1 \tag{4.2}$$

It follows that $h_0 = h_0(s \mid c)$ is "hydrodynamic" ($h_0 \in \operatorname{Ker} \Lambda$), and a "nonhydrodynamic" solution h_1 of (4.2) ($h_1 \perp \operatorname{Ker} \Lambda$) will be completely determined in terms of h_0. In fact, if $\mathbf{P}$ is the projector on Ker Λ, we have

$$\begin{aligned} &\mathbf{P}(v\, \partial_s h_0) = 0 \\ &h_1 = \Lambda^{-1}(v\, \partial_s h_0) = \partial_s h_0 \Lambda^{-1}(v) \end{aligned} \tag{4.3}$$

$\Lambda^{-1}(v)$ is explicitly computable (see Appendix).

The hydrodynamic equation for h_0 is of Navier–Stokes type; it comes from the first-order equation in the Chapman–Enskog expansion (the Euler equation is trivial):

$$\mathbf{P}(v\,\partial_s(h_0 + \varepsilon h_1)) = 0$$

Thus, by Eqs. (4.3) we obtain

$$\mathbf{P}(v\,\partial_s h_0) + \varepsilon \mathbf{P}(v\,\partial_s(\Lambda^{-1}(v)\,\partial_s h_0)) = 0$$

i.e.,

$$\mathbf{P}(v\Lambda^{-1}(v))\,\partial_s^2 h_0 = 0$$

This gives the macroscopic equation $\partial_s^2 h_0 = 0$, which, via the boundary conditions $h_0(\pm 1 \mid c) = \psi_\pm(c)$, is explicitly solvable:

$$h_0(s \mid c) = \frac{\psi_+(c) - \psi_-(c)}{2}\,s + \frac{\psi_+(c) + \psi_-(c)}{2}$$

$$h_1(v \mid c) = \frac{\psi_+(c) - \psi_-(c)}{2}\,\Lambda^{-1}(v)$$

so that

$$\|h_1\|_\infty < \text{const}$$

From Eq. (4.1) we find the equation for the error $R^\varepsilon := h^\varepsilon - (h_0 + \varepsilon h_1)$,

$$\begin{aligned} v\,\partial_s R^\varepsilon &= \varepsilon^{-1}\Lambda R^\varepsilon \\ R^\varepsilon(+1, v, c) &= -\varepsilon h_1(v \mid c), \qquad v < 0 \\ R^\varepsilon(-1, v, c) &= -\varepsilon h_1(v \mid c), \qquad v > 0 \end{aligned}$$

The estimate of the solution in term of the boundary data applies and we obtain

$$\|R^\varepsilon\|_\infty \leqslant \varepsilon\,\|h_1\|_\infty$$

Thus, we have

$$\|h^\varepsilon - h_0\|_\infty \leqslant 2\varepsilon\,\|h_1\|_\infty$$

this means

$$\|f^\varepsilon(s, v, c) - g(v)\,h_0(s \mid c)\|_\infty \leqslant g(v)\,\|h^\varepsilon - h_0\|_\infty \leqslant 2\varepsilon\,\|g\|_\infty\,\|h_1\|_\infty$$

and Theorem 2 is proven.

5. A DISCRETE VERSION OF THE MODEL

We shall calculate the explicit solution of a discretized version of the previous model, i.e., let us restrict the values of admissible v_x velocities to the set $\{-1, +1\}$.

The solution of this very simplified problem, which will be given in explicit form, shows the same qualitative behavior as the "full" solution as the size of the slab goes to infinity.

In this model the "color" is parametrized by the same continuous variable $c \in \mathbb{R}$, and $f_\pm(x, c)$ is the distribution function corresponding to $v_x = \pm 1$.

Again $g_\pm := \int f_\pm(x, c)\, dc$ do not depend on x, and the boundary conditions at $\pm L$ define $g_\pm$ up to a normalization constant.

The resulting system becomes ($x \in [-L, +L]$)

$$\partial_x f_+(x, c) = 2g_+ f_-(x, c) - 2g_- f_+(x, c)$$
$$-\partial_x f_-(x, c) = 2g_- f_+(x, c) - 2g_+ f_-(x, c)$$

with the boundary conditions

$$f_+(-L, c) = H_-(c)\, g_-, \qquad f_-(+L, c) = H_+(c)\, g_+$$

The condition of vanishing net flux gives, with $\int H_\pm\, dc = 1$, $g_- = g_+ := g$ (this number plays the role of a normalization constant).

Straightforward calculations [starting from $\partial_x(f_+ - f_-) = 0$] lead to the final form

$$f_\pm(x, c) = 2g^2 \frac{H_+(c) - H_-(c)}{1 + 4gL}(x \pm L) + H_\mp(c)\, g$$

We end with the asymptotic behavior in the scaled space variable $s = x/L = \varepsilon x$:

$$f^\varepsilon_\pm(s, c) = \tfrac{1}{2} g\{[H_+(c) + H_-(c)] + [H_+(c) - H_-(c)]s\}$$
$$+ \tfrac{1}{8}\varepsilon\{[H_-(c) - H_+(c)](s \pm 1)\} + o(\varepsilon)$$

We note that at the zeroth-order term in ε the two distributions f^ε_+ and f^ε_- coincide: in fact, this term describes the local equilibrium given by a global equilibrium term plus a linear term as in the general case.

APPENDIX

We want to show the explicit form of the solutions to the equation

$$\Lambda\phi(v) = v \tag{A.1}$$

where the operator Λ is defined in (3.3).

First, one can observe that the function on the right-hand side of (A.1) belongs to the range of Λ, being orthogonal to Ker Λ in $\mathscr{L}_2(\mathbb{R}, g(v)\,dv)$. This comes from the conservation of mass at the walls: $\int vg(v)\,dv = 0$.

Let us first see in more detail the structure of Ker Λ: if $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathscr{L}_2(\mathbb{R}, g(v)\,dv)$, from the analysis of the quadratic form associated with Λ we get

$$\phi_0 \in \text{Ker}\,\Lambda \Leftrightarrow \langle \phi_0, \Lambda\phi_0 \rangle = 0$$

On the other hand, for every $\psi \in D(\Lambda)$

$$\langle \psi, \Lambda\psi \rangle = -1/2 \int g(v)\, g(w)\, |v-w|\, [\psi(v) - \psi(w)]^2\, dw\, dv$$

which gives

$$\phi_0 \in \text{Ker}\,\Lambda \Leftrightarrow \phi_0(v) = \phi_0(w) = \text{const}$$

Let us now return to the previous question, noting that the (infinite) solutions of Eq. (A.1) will be defined up to arbitrary constants.

From the identity, valid for every $\psi \in D(\Lambda)$,

$$\left[\int |v-u|\, \psi(v)\, g(u)\, du\right]'' = 2g(v)\, \psi(v)$$

by double differentiation with respect to v of (A.1), we have

$$-2k'(v)\, \psi'(v) - k(v)\, \psi''(v) = 0$$

We easily get

$$\psi(v) = c_1 \int_z^v \frac{du}{k^2(u)}$$

where z and c_1 are to be chosen in the following; we need to find c_1 in order that

$$c_1 \int |v-w| \left[\int_v^w \frac{du}{k^2(u)}\right] g(w)\, dw = v$$

(where obviously the "additive" constant z does not appear, since Ker $\Lambda = \{\text{const.}\}$).

The choice of c_1 is easily made by looking to the asymptotic behavior as $v \to \infty$:

$$c_1 \int |v-w| \left[\int_v^w \frac{du}{k^2(u)} \right] g(w)\, dw = v$$

implies, as $v \to +\infty$,

$$c_1 \int g(w) \int_{+\infty}^{w} \frac{du}{k^2(u)}\, dw = 1 \Rightarrow c_1 = \left[\int g(w) \int_{+\infty}^{w} \frac{du}{k^2(u)}\, dw \right]^{-1}$$
$$= -2 \left[m_0 \int_{-\infty}^{+\infty} \frac{dv}{k^2(v)} \right]^{-1}$$

where $m_0 = \int g(v)\, dv$.

To get this result we used the identity $2g(v) = k''(v)$ and some integrations by parts.

Now we can choose the other constant z to obtain a solution ψ such that, if $z = z_0$,

$$\psi \perp \operatorname{Ker} \Lambda; \qquad \text{i.e.,} \quad \langle 1, \psi \rangle = \int g(v) \int_{z_0}^{v} \frac{du}{k^2(u)}\, dv = 0$$

This choice of z_0 is easily seen to be possible and unique, because the scalar product $\langle 1, \psi \rangle$ depends on z in a monotonic way, and changes sign when z varies from $-\infty$ to $+\infty$. So we can write the "nonhydrodynamic" solution ϕ_0 to Eq. (A.1) in the following explicit form:

$$\phi_0(v) = -2 \left[m_0 \int_{-\infty}^{+\infty} \frac{dv}{k^2(v)} \right]^{-1} \int_{z_0}^{v} \frac{dw}{k^2(w)}$$

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NOTE ADDED IN PROOF

A more general uniqueness result is obtained by using the energy estimate for the homogeneous problem: scalar multiplying the equation with the unknown itself and integrating on $x \in [-L, L]$, from the nonpositivity of Λ and homogeneity of boundary conditions, we get the uniqueness result.

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